Scattering in Euclidean space: from Bethe-Salpeter to LQCD

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Prelude

Despite that physical space-time has a Minkowski structure, the interest in formulating a relativistic theory (QFT) in an Euclidean metric was already present in the very early days

It was first applied by G. Wick to solve the Bethe-Salpeter equation: "Wick rotation" $k_0 = i\mathbf{k_4}$

$$k^2 = k_0^2 - \vec{k}^2 = -(k_4^2 + \vec{k}^2) \equiv -k_E^2$$

G. Wick, Properties of the Bethe-Salpeter wave functions, Phys. Rev. 96 (1954) 1124

and formulated in a more general QFT framework soon later

J. Schwinger

On the Euclidean structure of relativistic field theory, Proc. Natl. Acad. Sci. U.S.A 44 (1958) 95

Euclidean Quantum Electrodynamics, Phys Rev (1959) 721

K. Symanzik

Euclidean Quantum Field Theory I. Equations for a scalar model, J. Math. Phys. 7 (1966) 510

Euclidean Quantum Field Theory in Local Quantum Field Theory, Ed. by R. Jost, Academic Press Ne York (1969)

International School of Physics Enrico Fermi, Course XLV, Ed. by R. Jost (1968)

This interest was enhanced by the success of **Lattice** calculations, where it is **mandatory !**

The main theoretical problem is to prove the equivalence between both formulations

K. Osterwalder and E. Schrader, Axioms for Euclidean QFT, Comm. Math. Phys. 42 (1975) 440

Gave conditions for a QFT to ensure that the "analytic continuation" of Green functions can be safely done (no singularities) …but they are not yet proved for theories of interest

Prelude

The **practical benefit** of Minkowski -> Euclidean **is clear** :

Everything becomes smooth (allowing standard methods for solving integral equations)

Euclidean actions S_E becomes positive definite (allowing path integrals)

……

Much less clear is what is lost when using an Euclidean formulation…

I Form factors $F(Q^2)$ in time-like region are not allowed $Q^2 = Q_0^2 - \vec{q}^2 = -(Q_4^2 + \vec{q}^2) \equiv -Q_E^2$

It is even not clear that the space-like are correctly computed

II Scattering is lost:

Maiani-Testa "no go theorem" PLB 245 (1990) 585: no scattering in **infinite** euclidean space

However circunvented in LQCD finite volume by "Luscher method"

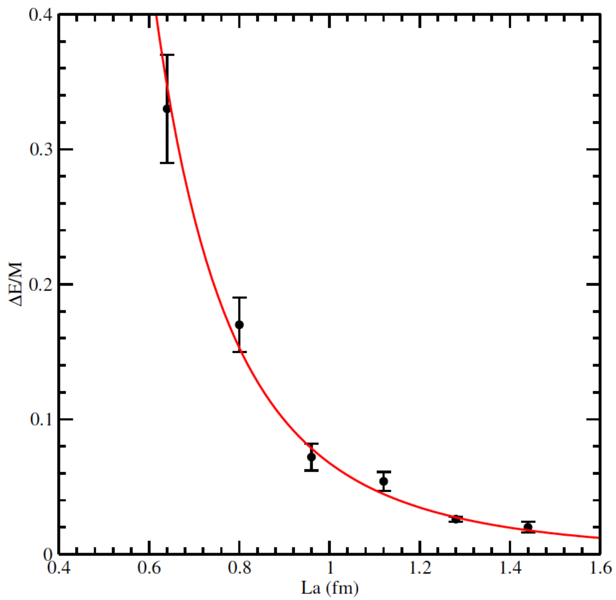
III Problems for introducing chemical potential (at T>0)

….

Luscher method (87)

Energy levels $\varepsilon_n(L)$ of 2-particle in a periodic box (L) provide phase-shifts

In the simplest case: ground state $\varepsilon_0(L)$ provides the scattering length A_0



$$\epsilon_0(L) = \frac{4\pi A_0}{(aL)^3}\left\{1 + c_1\left(\frac{A_0}{aL}\right) + c_2\left(\frac{A_0}{aL}\right)^2 + \dots\right\}$$

c_1 and c_2 are universal known coefficients

More involved expressions provide the phase shifts $$k\cot\delta_0(k) = \frac{1}{\pi a L}\,S(\eta)$$

It works well ! …but has problems with open channels and for large lattices

Prelude

General results are sometimes too deep to be clear….

Bethe-Salpeter (BS) framework is an ideal landscape to have some light on this problem

I

Formulated in terms of BS amplitude which - contrary to wave functions – have a clear definition in terms of QFT and is accessible to LQCD

II

It fulfills a 4D integral equation which - in its (simplified) version – can be solved both in Euclidean and Minkowski space

Aim of this talk :

I **Present a method for obtaining BS scattering solutions in Minkowski space**

II **Discuss the problem with Euclidean BS scattering solutions**

III **Obtain a purely Euclidean equation for zero energy (scattering length)**
Alternative to Luscher method in Lattice calculations

IV **Present possible extension to non zero energies** (Effective Range approximation)

Introduction

Bethe Salpeter equation deals with a - pre-existing - QFT object (Gell-Mann Low)

$$\Phi(x_1, x_2, P) =< 0 \mid T\{\phi(x_1)\phi(x_2)\} \mid P >$$

Its Fourier transform Φ(k,P)

$$\Phi(x_1, x_2, P) = \int \frac{dp_1}{(2\pi)^4} \frac{dp_2}{(2\pi)^4} \Phi(p_1, p_2)\, e^{-iPx}\, e^{-ikx} = e^{-iPx} \int \frac{dk}{(2\pi)^4} \Phi(k, P)\, e^{-ikx}$$

$$\begin{aligned} p_1 + p_2 &= P \\ p_1 - p_2 &= 2k \end{aligned}$$

satisfies a 4D equation. For bound state case:

$$\boxed{\Phi(k, P) = S_1(k, P) S_2(k, P) \int \frac{d^4k'}{(2\pi)^4}\, iK(k, k'; P)\, \Phi(k', P)} \qquad \textbf{(1)}$$

P^2=M^2 with M the total mass of the two-body system

S_i = free propagators

$$\begin{aligned} \mathbf{S_1(k, P)} &= \frac{\mathbf{i}}{\left(\frac{\mathbf{P}}{2} + \mathbf{k}\right)^2 - \mathbf{m^2} + \mathbf{i}\epsilon} \\ \mathbf{S_2(k, P)} &= \frac{\mathbf{i}}{\left(\frac{\mathbf{P}}{2} - \mathbf{k}\right)^2 - \mathbf{m^2} + \mathbf{i}\epsilon} \end{aligned}$$

iK=Interaction kernel

- if K would contain all the IR graphs, solving (1) would be equivalent to solve the full QFT
- This is however a wishful thinking. In practice one uses a very poor restriction: ladder+simple kernels

It is usually written in terms of the “vertex function” $F(k,p) = \dfrac{\Phi(k,p)}{S_1(k,p)S_2(k,p)}$

That is $$F(k;p) = \int \frac{d^4k'}{(2\pi)^4} \frac{iK(k,k')F(k';p)}{\left[\left(\frac{p}{2}+k'\right)^2 - m^2 + i\epsilon\right]\left[\left(\frac{p}{2}-k'\right)^2 - m^2 + i\epsilon\right]}$$

As one can see the equation has singularities in the free propagators

As well as in the simplest (one boson exchange ladder) kernel

$$K(k,k) = -\frac{g^2}{(k-k')^2 - \mu^2 + i\epsilon}$$

Bound-state Solutions in Euclidean space

Bound states are easily solved with the "**Wick rotation**" $k_0 = i\mathbf{k_4}$

Minkowski $(k_0,k) \rightarrow$ Euclidean (k_4,k)

$$k^2 = k_0^2 - \vec{k}^2 = -(k_4^2 + \vec{k}^2) = -k_E^2$$

It leads to a smooth **integral equation for the euclidean amplitude** $\Phi_E(k_4,\vec{k}) \equiv \Phi_M(k_0 = ik_4,\vec{k})$

$$\left[\left(k_4^2 + k^2 + m^2 - \frac{M^2}{4}\right)^2 + M^2 k_4^2\right]\Phi_E(k_4,\vec{k}) = g^2 \int \frac{dk'_4 d\vec{k}'}{(2\pi)^4}\,\frac{1}{(k-k')_E^2 + \mu^2}\,\Phi_E(k'_4,\vec{k}')$$

soluble by standard methods.

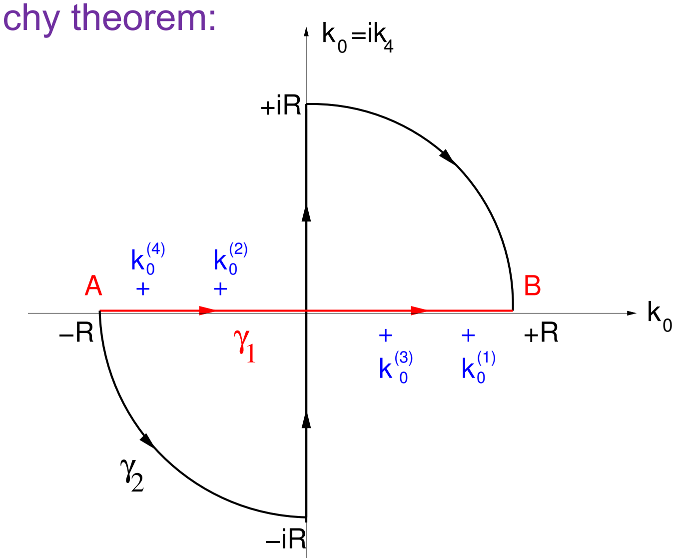
Until very recently, most of the existing solutions were found in this way

Rm: Not a simple « variable change » but a straightforward application of Cauchy theorem:

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

If a continous deformation between two paths do not cross a singularity of f(z)

Ex: free propagators



There are however some problems:

- The validity of the Wick rotation requires a careful analysis of kernel singularities
  It is done only in some cases (ladder WC)

- The total mass M is invariant but the BS amplitude is not: impossible to recover Φ_M from Φ_E

- What about other observables: scattering amplitudes, form factors, …
  only accessible in Minkowski space!

All these reasons motivated a series of works for obtaining the Minkowski BS solutions

I Compute scattering observable
II Compute form factors

We developed two totally independent methods

- Light-Front projection of the BS equation and Nakanishi representation of the amplitude

- A « direct » approach (*)

(*) J.Carbonell and V.A. Karmanov Phys. Lett. B 727, 319 (2013), Phys. Rev. D 90, 056002 (2014)

Minkowski space solutions : direct method

In terms of "Vertex function" $$F(k,p) = \frac{\Phi(k,p)}{S_1(k,p)S_2(k,p)}$$

and for the scattering process $k_{1s} + k_{2s} \to k_1 + k_2$ the BS equation reads

$$F(k;k_s) = K(k,k_s) - i\int \frac{d^4k'}{(2\pi)^4} \frac{K(k,k')F(k';k_s)}{\left[\left(\frac{p}{2}+k'\right)^2 - m^2 + i\epsilon\right]\left[\left(\frac{p}{2}-k'\right)^2 - m^2 + i\epsilon\right]} \qquad \textbf{(2)}$$

In Minkowski space, **this equation is plagued with singularities** from 4 different sources :

1. The singular inhomogeneous **Born term**

2. The 2+2 poles of the constituent **propagators**

$$\frac{1}{p'_0 - a - i\epsilon} = PV\left\{\frac{1}{p'_0 - a}\right\} + i\pi\delta(p'_0 - a)$$

2. The **kernel** singularities

4. The singular behaviour of **F itself** (due to K, but not only)

A careful analysis of all these singularities allows (*) to obtain the full BS (off-mass shell) scattering amplitude, even above the inelastic thresholds ($N_1+N_2 \to N_1+N_2+m+..$).
For **the first time since its formulation …in 54 !**

(*) J. Carbonell and V.A. Karmanov Phys. Lett. B 727, 319 (2013), Phys. Rev. D 90, 056002 (2014)

Minkowski espace solutions : direct method

After many steps (*) the S-wave BS equation in Minwoski space takes the form

$$
\begin{aligned}
F_0(k_0,k) &= F_0^B(k_0,k) \\
&+ \frac{i\pi^2 k_s}{8\varepsilon_{k_s}} W_0^S(k_0,k,0,k_s)F_0(0,k_s) \\
&+ \frac{\pi}{2M}\int_0^\infty \frac{dk'}{\varepsilon_{k'}(2\varepsilon_{k'}-M)}\left[k'^2 W_0^S(k_0,k,a_-,k')F_0(|a_-|,k') - \frac{2{k_s}^2\varepsilon_{k'}}{\varepsilon_{k'}+\varepsilon_{k_s}}W_0^S(k_0,k,0,k_s)F_0(0,k_s)\right] \\
&- \frac{\pi}{2M}\int_0^\infty \frac{k'^2 dk'}{\varepsilon_{k'}(2\varepsilon_{k'}+M)}W_0^S(k_0,k,a_+,k')F_0(a_+,k') \\
&+ \frac{i}{2M}\int_0^\infty \frac{k'^2 dk'}{\varepsilon_{k'}}\int_0^\infty dk_0' \left[\frac{W_0^S(k_0,k,k_0',k')F_0(k_0',k') - W_0^S(k_0,k,a_-,k')F_0(|a_-|,k')}{k_0'^2-a_-^2}\right] \\
&- \frac{i}{2M}\int_0^\infty \frac{k'^2 dk'}{\varepsilon_{k'}}\int_0^\infty dk_0' \left[\frac{W_0^S(k_0,k,k_0',k')F_0(k_0',k') - W_0^S(k_0,k,a_+,k')F_0(a_+,k')}{k_0'^2-a_+^2}\right]
\end{aligned}
$$

$$a_\pm(k',k_s) = \varepsilon_{k'} \pm \varepsilon_{k_s}$$

In this form all the PV coming from propagator poles have been absorbed (by sustraction)
It remains « only » to treat the singularities of the kernel W_0^S, of the Born term F^B … and solve it !

Quite a nasty equation for an S-wave… compared to the NR Lipmann-Schwinger one
« c'est la vie » in Minkowski space !

(*) J.C. and V.A. Kamanov, Phys Rev D90 (2014) 056002

This is however the only way to compute in the whole kinematical domain:

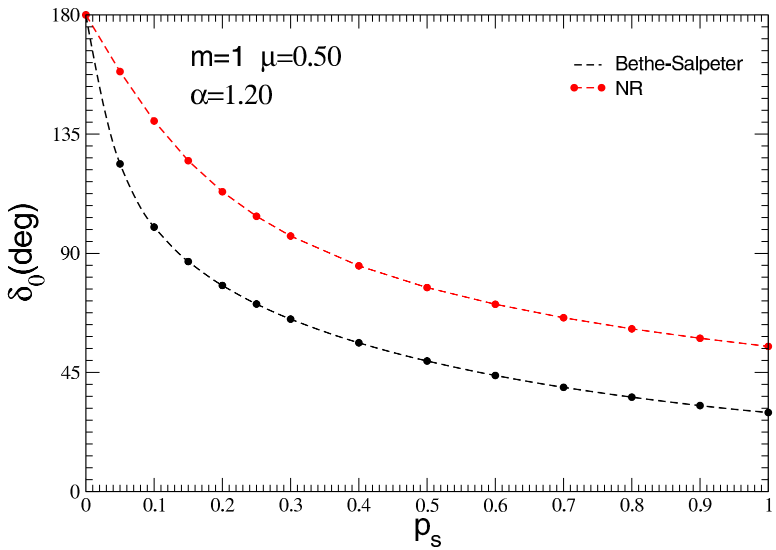
- the off-mass shell $F(k,k_s;p)$
- the half off-mass shell $F(k,k_s)$ (with p related to k_s)

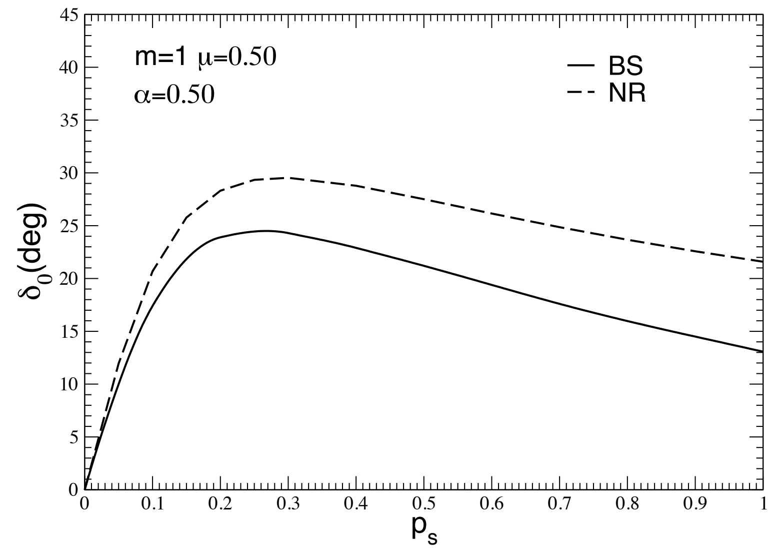
and solve the BS scattering problem in its full complexity (including inelastic thresholds)

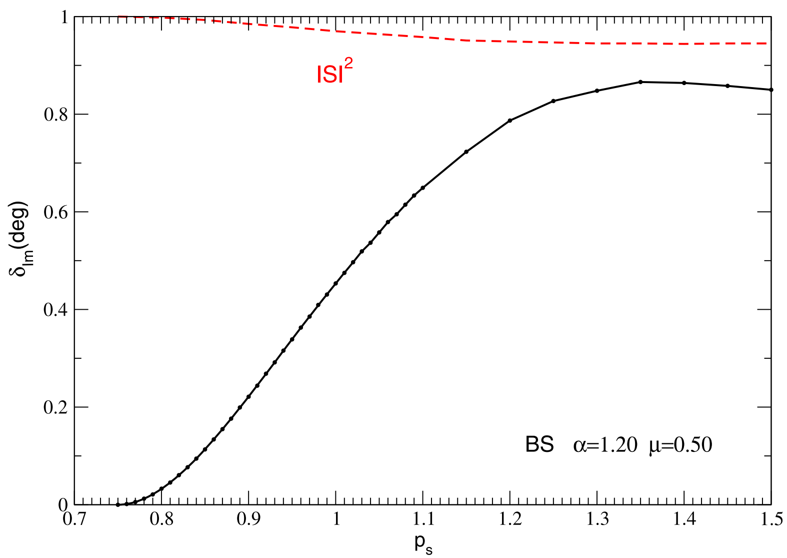
The observables are obtained from the on-shell value $F^{on}_0(k_0=0,k=k_s)$

e.g. the phase shifts $\delta_l = \frac{1}{2i}\log\Big(1+\frac{2ip_s}{\varepsilon_{p_s}}F_l^{on}\Big)$

Phase shifts and inelasticities





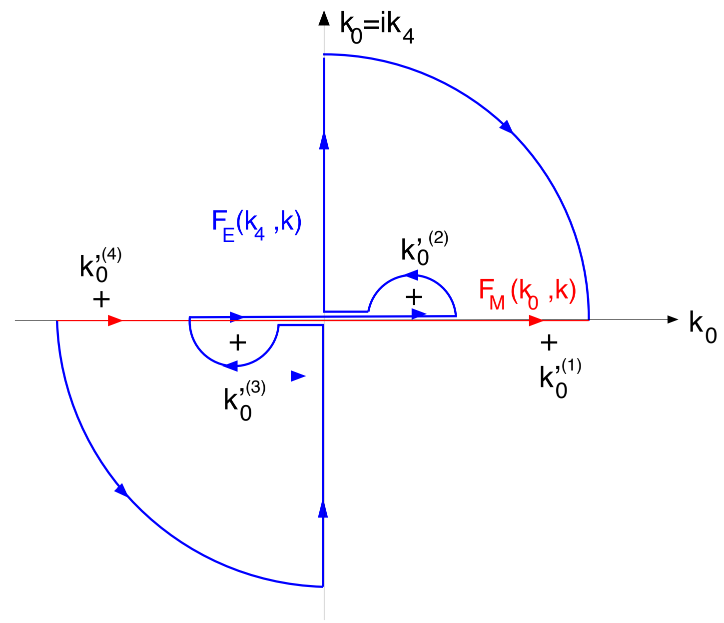


The problem with the Euclidean espace solutions

Assume we want to obtain a scattering BS equation for the euclidean amplitude $F_E(k_4,k)=F_M(k_0=ik_4,k)$ by **properly applying the Wick rotation**, i.e. taking into account the singularities

$$\begin{aligned}
{k'}_0^{(1)}(k,k_s) &= \varepsilon_{k_s}+\varepsilon_{k'}-i\epsilon=+a_+-i\epsilon\\
{k'}_0^{(2)}(k,k_s) &= \varepsilon_{k_s}-\varepsilon_{k'}+i\epsilon=-a_-+i\epsilon\\
{k'}_0^{(3)}(k,k_s) &= -\varepsilon_{k_s}+\varepsilon_{k'}-i\epsilon=+a_--i\epsilon\\
{k'}_0^{(4)}(k,k_s) &= -\varepsilon_{k_s}-\varepsilon_{k'}+i\epsilon=-a_++i\epsilon
\end{aligned}$$

$$a_\pm(k',k_s)=\varepsilon_{k'}\pm\varepsilon_{k_s}$$



The initial BS equation **(2)** becomes

$$F^E(k_4,\vec{k};\vec{k}_s)=V^B(k_4,\vec{k};\vec{k}_s)+\int\frac{d^4k'}{(2\pi)^4}\frac{V(k_4,\vec{k'};k'_4,\vec{k'})F^E(k'_4,\vec{k'};\vec{k}_s)}{({k'_4}^2+a_-^2)({k'_4}^2+a_+^2)}+S(k_4,k,k_s) \qquad \textbf{(E1)}$$

with « rotated » kernels

$$V(k_4,\vec{k};k'_4,\vec{k'})=\frac{16\pi m^2\alpha}{(k_4-k'_4)^2+(\vec{k}-\vec{k'})^2+\mu^2};$$

$$V^B(k_4,\vec{k};\vec{k}_s)=V(k_4,\vec{k};k'_4=0,\vec{k'}=\vec{k}_s)$$

Pole contribution results into a term (S) mixing F_E and F_M : **equation for F_E alone is impossible !!!**

The (integral) term S contains the Minkoswki amplitude at the particular value $\mathbf{F_M(k_0 = \epsilon_{k_s} - \epsilon_k, k)}$

In addition to **(E1)**, a second integral equation is needed to solve the problem !
One ends with a system of two-coupled equations involving both $\mathbf{F_E}$ and $\mathbf{F_M}$ denoted symbolically

$$\begin{aligned} F_E(k_4,k) &= \mathcal{I}_1\left[F_E(k'_4,k'),F_M(\epsilon_{k_s}-\epsilon_{k'},k')\right] && \textbf{(E1)} \\ F_M(\epsilon_{k_s}-\epsilon_k,k) &= \mathcal{I}_2\left[F_E(k'_4,k'),F_M(\epsilon_{k_s}-\epsilon_{k'},k')\right] && \textbf{(E2)} \end{aligned}$$

This system of equations was first obtained in **M. Levin, J. Wright, and J. Tjon, Phys Rev 254 (1967) 1433**
It was derived independently in **J.C. and V.A. Kamanov, Phys Rev D90 (2014) 056002**
where it was used to check the « direct » solution F_M (for the particular value of k_0)

R1: Equations **(E1+E2)** remain singular (in the Minkowski part)

R2: They do not provide the amplitude F_M in the full (k_0,k) plane but in one-dimensional manifold (string)

R3: On-mass shell, i.e. k_0=0, one has $F_M(0,k_s) = F_E(0,k_s)$ and the solution provides elastic phase shifts

The system of equations (E1+E2) coupling F_E and F_M shows the impossibility to solve the BS scattering problem using an eucliden metric

.....Always ???

The case of zero energy scattering (k_s=0)

It can be shown(*) that the additional term S coupling to the Minkowski vanishes in the limit k_s=0

One obtain this way a **regular purely Euclidean equation for F_E**. For S-waves it reads

$$F_E(k_4,k) = V^B(k_4,k) + \int_0^\infty k'^2 dk' \int_0^\infty dk'_4 \, \frac{V_0(k_4,k;k'_4,k')}{({k'_4}^2+{a'_-}^2)({k'_4}^2+{a'_+}^2)} \, F_E(k'_4,k')$$

The scattering length is directly given by **a_0=-F_E(0,0)/m**

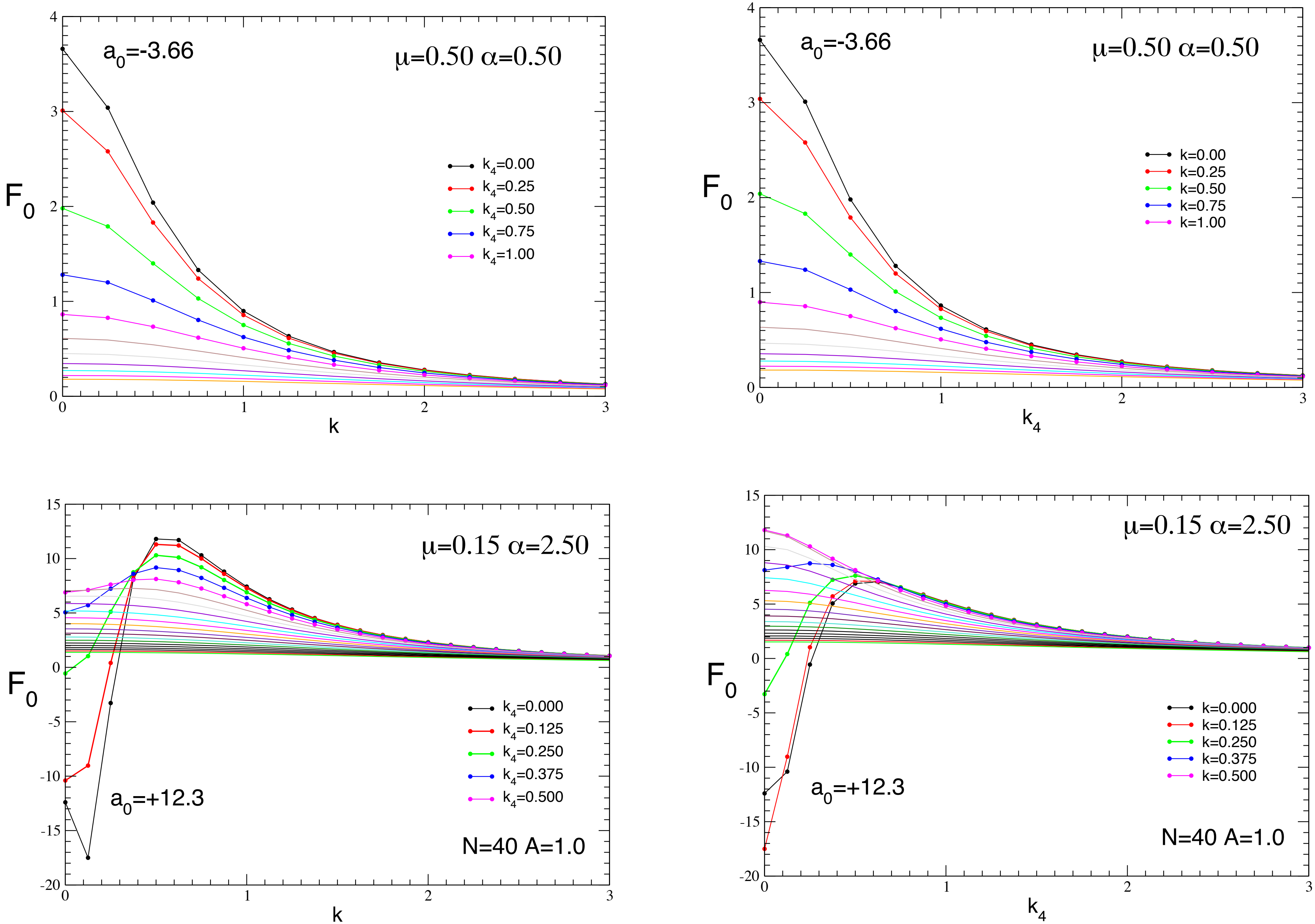
Apart from providing a very stable and cheap scattering length in BS equation (like bound states) it demonstrates, in the BS framework, the possibility to obtain (till now zero energy!) **scattering results from purey Euclidean solutions**

This suggests an **alternative way to Luscher method** for computing scattering observables in Lattice calculations : one needs only (the Fourier transform of) the Euclidean version of

$$\Phi(x_1,x_2,P) = < 0 \mid T\{\phi(x_1)\phi(x_2)\} \mid P >$$

This quantity has been computed since many years in LQCD collaborations (see Ikeda san talk))
but never used to compute a_0
Could you please try ???

(*) J.C. and V.A. Kamanov, Phys. Lett. B754 (2016) 270



a0=-3.66
μ=0.50 α=0.50
F0
k
k4=0.00
k4=0.25
k4=0.50
k4=0.75
k4=1.00
a0=-3.66
μ=0.50 α=0.50
F0
k4
k=0.00
k=0.25
k=0.50
k=0.75
k=1.00
μ=0.15 α=2.50
F0
k
k4=0.000
k4=0.125
k4=0.250
k4=0.375
k4=0.500
a0=+12.3
N=40 A=1.0
μ=0.15 α=2.50
F0
k4
k=0.000
k=0.125
k=0.250
k=0.375
k=0.500
a0=+12.3
N=40 A=1.0

Conclusion

We have presented a method for solving Bethe-Salpeter equation in Minkowski space.
based on a **direct solution** with a careful analysis of singularities

We have shown that **Minkowski solutions are mandatory** for the **scattering problem**

We propose a new method to compute the **scattering length** (zero scattering energy) from a purely **Euclidean BS amplitude F_E** (vertex function)
It provides an **alternative** to Luscher method used **in Lattice** calculations
and it is extensible to non zero energy in the effective range approximation.